PMM U.S.S.R., Vol. 46, No. 5, pp. 571-576, 1983.
Printed in Great Britain.

0021-8928/83 $7.50+0.00

UDC 62-50

BALANCED COALITION COUNTERSTRATEGIES IN MANY-PERSON DIFFERENTIAL GAMES*

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An m-person positional differential game whose dynamics are described by a nonlinear differential equation is analyzed. The following information hierarchy is assumed to holds: each player knows the controls, realized at the current instant, of the players numbered higher than himself. The concept of a balanced coalition counterstrategy of the m players is introduced and an existence theorem is proved for it. A method for constructing the balanced coalition counterstrategy is indicated. The results are presented of a numerical example describing the planar motion of a material point under forces commanded by three players. The paper is closely related to the researches in /1 − 5/.

1. We consider an m-person differential game whose dynamics are described by the nonlinear differential equation

$$x^{\cdot} = f(t, x, u_1, \ldots, u_m) \tag{1.1}$$

where x is an n-dimensional vector, u_i is the i-th player's vector-valued control subject to the constraint $u_i \in P_i$, P_i is a compactum in space R^{k_i}. The function f is continuous in all the variables and satisfies a Lipschitz condition with respect to x in the domain $G \times P_1 \times \ldots \times P_m$. Here G is a compactum in space $\{t, x\}$ with a projection onto the t-axis equal to a prescribed interval $[t_0, T]$; we assume that any trajectory of system (1.1), starting off in G, remains in G up to the instant T. The i-th player's purpose is to choose the control u_i so as to minimize the quantity $\sigma_i(x[T])$, where $\sigma_i : R^n \mapsto R^1$ are prescribed continuous functions, as system (1.1) goes from an arbitrary initial position $(t_*, x_*) \in G$ into a state $x[T]$ at instant $t = T$.

We denote $M = \{0, 1, \ldots, m\}$. Let $t_* \in [t_0, T]$. We say that a piecewise-constant right-continuous function $\alpha : [t_*, T] \mapsto M$ is of class $A_m[t_*, T]$ if it has at most one point of discontinuity (subsequently denoted ϑ) and then only under the condition that $\alpha[t_*] = 0$. Thus, the class $A_m[t_*, T]$ consists of functions $\alpha^{\circ}[t]$ identically zero on $[t_*, T]$, of functions $\alpha_{\vartheta}^{i}[t]$, $i = 1, \ldots, m$, $t_* \leqslant \vartheta < T$ equal to zero on $[t_*, \vartheta)$ and to i on $[\vartheta, T]$, and, finally, of functions $\alpha_T^{i}[t]$ which we identify with the functions $\alpha^{\circ}[t]$. We assume that at the current instant t each of the m players knows the system's phase vector $x[t]$ as well as the value of some functions from class $A_m[t_*, \vartheta]$. In addition, the following information hierarchy exists in the system: a player numbered i $(i = 1, \ldots, m)$ knows the controls realized at the current instant by the players numbered $i + 1, \ldots, m$.

Definition 1.1. A mapping $h_\alpha = h_\alpha(t, x, u_2, \ldots, u_m, \varepsilon)$ defined for $(t, x) \in G$, $u_2 \in P_2, \ldots, u_m \in P_m$, $\varepsilon > 0$, $\alpha \in M$, such that

$$\begin{aligned} &h_0(t, x, u_2, \ldots, u_m, \varepsilon) = \{u_1^{(0)}(t, x, u_2, \ldots, u_m, \varepsilon), u_2^{(0)}(t, x, \\ &u_3, \ldots, u_m, \varepsilon), \ldots, u_{m-1}^{(0)}(t, x, u_m, \varepsilon), u_m^{(0)}(t, x, \varepsilon)\} \\ &h_i(t, x, u_2, \ldots, u_m, \varepsilon) = \{u_1^{(i)}(t, x, u_2, \ldots, u_m, \varepsilon), \ldots, \\ &\quad u_{i-1}^{(i)}(t, x, u_i, \ldots, u_m, \varepsilon), u_{i+1}^{(i)}(t, x, u_{i+2}, \ldots, u_m, \varepsilon), \ldots, \\ &\quad u_{m-1}^{(i)}(t, x, u_m, \varepsilon), u_m^{(i)}(t, x, \varepsilon)\}, \; i = 1, \ldots, m \end{aligned} \tag{1.2}$$

where

$$u_1^{(\alpha)}(t, x, u_2, \ldots, u_m, \varepsilon), u_2^{(\alpha)}(t, x, u_3, \ldots, u_m, \varepsilon), \ldots, u_{m-1}^{(\alpha)}(t, x, u_m, \varepsilon), u_m^{(\alpha)}(t, x, \varepsilon), \alpha \in M$$

are functions, Borel-measurable in the variables $u_2, \ldots, u_m$, with values in $P_1, P_2, \ldots, P_{m-1}, P_m$, respectively, is called a coalition counterstrategy (c.c.s.) of the m players. Thus, for a fixed α the mapping h_α associates with the collection $(t, x, u_2, \ldots, u_m, \varepsilon)$ a collection of controls of form (1.2) of the m players; and, for $\alpha = 1, 2, \ldots, m$ the control of the player numbered α does not occur the corresponding collection. We denote the c.c.s. by the symbol $H \div h_\alpha(t, x, u_2, \ldots, u_m, \varepsilon)$. A meaningful interpretation of the concept of a c.c.s. is given below.

*Prikl.Matem.Mekhan.,46,No.5,pp.714-721,1982

Let there be given the c.c.s. $H \div h_\alpha(t, x, u_2, \ldots, u_m, \varepsilon)$, some initial position $(t_*, x_*) \in G$, a partitioning Δ of interval $[t_*, T]$ into a system of nonintersecting half-open intervals $[\tau_j, \tau_{j+1})$; the partitioning's step is denoted $\delta(\Delta) = \max_j (\tau_{j+1} - \tau_j)$. We assume that the function $\alpha_\vartheta^i[\cdot] \in A_m[t_*, T]$ is realized during the process. If the point ϑ does not coincide with even one of the nodes τ_j of partitioning Δ, then it is taken as an additional node in this partitioning. We define the Euler polygonal line generated by the c.c.s. H under partitioning Δ and realization $\alpha_\vartheta^i[\cdot]$ thus; it is the absolutely continuous function $x_\Delta^\varepsilon[t] = x_{\Delta^i}[t, t_*, x_*, H, \alpha_\vartheta^i, u_i^*[\cdot]]$, $x_\Delta^\varepsilon[t_*] = x_*$, satisfying the equation

$$\begin{aligned} &x_\Delta^{\cdot\varepsilon}[t] = f(t, x_\Delta^\varepsilon[t], v_1^{(0)}, v_2^{(0)}, \ldots, v_{m-1}^{(0)}, v_m^{(0)}, \varepsilon) \\ &t \in [\tau_j, \tau_{j+1}), \quad j = 0, 1, \ldots, s, \ \vartheta = \tau_{s+1} \\ &v_m^{(0)} = u_m^{(0)}(\tau_j, x_\Delta^\varepsilon[\tau_j], \varepsilon) \\ &v_{m-1}^{(0)} = u_{m-1}^{(0)}(\tau_j, x_\Delta^\varepsilon[\tau_j], v_m^{(0)}, \varepsilon), \ldots, \\ &v_1^{(0)} = u_1^{(0)}(\tau_j, x_\Delta^\varepsilon[\tau_j], v_2^{(0)}, \ldots, v_m^{(0)}, \varepsilon) \end{aligned} \tag{1.3}$$

and the equation

$$x_\Delta^{\cdot\varepsilon}[t] = f(t, x_\Delta^\varepsilon[t], v_1^{(i)*}[t], \ldots, v_{i-1}^{(i)*}[t], u_i^*[t], v_{i+1}^{(i)}, \ldots, v_{m-1}^{(i)}, v_m^{(i)}, \varepsilon) \tag{1.4}$$

for a.a. $t \in [\tau_j, \tau_{j+1})$, $j = s+1 \ldots$

(almost all)

$$\begin{aligned} &v_m^{(i)} = u_m^{(i)}(\tau_j, x_\Delta^\varepsilon[\tau_j], \varepsilon), \ldots, \\ &v_{i+1}^{(i)} = u_{i+1}^{(i)}(\tau_j, x_\Delta^\varepsilon[\tau_j], v_{i+2}^{(i)}, \ldots, v_m^{(i)}, \varepsilon), \\ &v_{i-1}^{(i)*}[t] = u_{i-1}^{(i)}(\tau_j, x_\Delta^\varepsilon[\tau_j], u_i^*[t], v_{i+1}^{(i)}, \ldots, v_m^{(i)}, \varepsilon), \ldots, \\ &v_1^{(i)*}[t] = u_1^{(i)}(\tau_j, x_\Delta^\varepsilon[\tau_j], v_2^{(i)*}[t], \ldots, v_{i-1}^{(i)*}[t], u_i^*[t], v_{i+1}^{(i)}, \ldots, v_m^{(i)}, \varepsilon) \end{aligned}$$

where $u_i^*[t]$ is an arbitrary measurable functions chosen as the control by the i-th player; $u_i^*[t] \in P_i$ for a.a. $t \in [t_*, T]$. We remark that to form the control $u_i^*[t]$ the i-th player can utilize the information on the game's current position and on the controls chosen by the players numbered $i+1, \ldots, m$. If the function $\alpha^0[\cdot]$ is realized during the game's progress, then the corresponding Euler polygonal line $x_\Delta^\varepsilon[t] = x_\Delta^\varepsilon[t, t_*, x_*, H, \alpha^0]$ satisfies the Eq. (1.3) on the whole interval $[t_*, T]$. The continuous function $x[t] = x[t, t_*, x_*, H, \alpha_\vartheta^i]$ specified by an iterated limit for every uniformly convergent double sequence of Euler polygonal lines

$$\begin{aligned} &x[t] = \lim_{k\to\infty} \lim_{l\to\infty} x_{\Delta_l^k}^{\varepsilon_k}[t, t_l^k, x_l^k, H, \alpha_{\vartheta_l^k}^i, u_{il}^{*k}[\cdot]] \\ &\lim_{l\to\infty} t_l^k = t_*, \quad \lim_{l\to\infty} x_l^k = x_*, \quad \lim_{l\to\infty} \vartheta_l^k = \vartheta \\ &\lim_{l\to\infty} \delta(\Delta_l^k) = 0, \quad \lim_{k\to\infty} \varepsilon_k = 0 \end{aligned} \tag{1.5}$$

is called the motion generated by the c.c.s. H and by the corresponding realization $\alpha_\vartheta^i[\cdot]$. Such a definition of motion in an m-person differential game as the iterated limit of a double sequence of Euler polygonal lines is similar to the definition of motion for an antagonistic game, introduced in /2/. The set of motions $x[t, t_*, x_*, H, \alpha_\vartheta^i]$ is denoted by $X(t_*, x_*, H, \alpha_\vartheta^i)$. The motions $x[t, t_*, x_*, H, \alpha^0]$ and the set $X(t_*, x_*, H, \alpha^0)$ are defined analogously.

Definition 1.2. A c.c.s. $H^p \div h_\alpha^p(t, x, u_2, \ldots, u_m, \varepsilon)$ is said to be balanced if the inequality

$$\min_{x[\cdot]} \sigma_i(x[T, \vartheta, \xi, H^p, \alpha_\vartheta^i]) \geqslant \max_{x[\cdot]} \sigma_i(x[T, \vartheta, \xi, H^p, \alpha^0]) \tag{1.6}$$

is valid for any position $(\vartheta, \xi) \in G$ and for any number $i = 1, 2, \ldots, m$.

A meaningful interpretation of the definition given is as follows. Let all m players, when forming their own controls, arrange to confine themselves to a collection of counter-strategies, prescribed by the mapping $h_0^p(t, x, u_2, \ldots, u_m, \varepsilon)$ (see (1.2)). Some coordinating body, located outside the system, monitors the implementation of the arrangement. We assume that at most one among the players can prove to be a violator of the arrangement. If all players have observed the arrangement up to the current instant $t \in [t_*, T]$, then at the instant t the coordinating body cummunicates to them the number zero (the value $\alpha[t] = 0$) signifying that there were no violaters. However, if at the instant $\vartheta \in [t_*, T)$ the i-th player violated the arrangement, then beginning with the instant ϑ the violator's index i is communicated to the players (the value $\alpha[t] = i$). By the same token, depending on whether or not there is a violation of the arrangement during the game, the coordinating body communicates to the players either one of the functions $\alpha_\vartheta^i[\cdot]$ or the function $\alpha^0[\cdot]$. Then inequality (1.6) signifies that the violating player obtains for himself a result which, in general, is no better than that which he could have guaranteed himself at the position (ϑ, ξ) in which the system arrived at the instant of violation, if all players had adhered to the arrangement. We remark that in the given setting the i-th player-violator has the possibility when forming the controls of utilizing the information on the controls of players numbered $i+1, \ldots, m$. However, a problem setting also is possible when the player-violator is deprived either partially or completely of the information on the other players' controls.

We note the constructive nature of the definition of a balanced c.c.s.. To be precise, for a balanced c.c.s. H^p, for a fixed $\varepsilon > 0$ and for any $\eta > 0$ we can find $\delta^*(\varepsilon, \eta) > 0$, $\zeta(\varepsilon, \eta) > 0$ such that for any position $(\vartheta, \xi) \in G$ and any $i = 1, 2, \ldots, m$ the inequalities

$$\inf_{x_\Delta^\varepsilon[\cdot]} \sigma_i(x_\Delta^\varepsilon[T, \vartheta^*, \xi^*, H^p, \alpha^i_{\vartheta_*}, u_i^*[\cdot]]) \geqslant \sup_{x_\Delta^\varepsilon[\cdot]} \sigma_i(x_\Delta^\varepsilon[T, \vartheta^{**}, \xi^{**}, H^p, \alpha^\circ]) - \eta \tag{1.7}$$

are valid for Euler polygonal lines with partitioning step $\delta(\Delta) < \delta^*$, for any admissible controls $u_i^*[\cdot]$, while

$$|\vartheta^* - \vartheta| < \zeta, \quad |\vartheta^{**} - \vartheta| < \zeta, \quad |\vartheta_* - \vartheta| < \zeta, \quad \|\xi^* - \xi\| < \zeta, \quad \|\xi^{**} - \xi\| < \zeta, \quad \vartheta_* \in [\vartheta^*, T).$$

The validity of this statement follows from the definition of the motions generated by c.c.s.

Theorem 1.1. In the m-person differential game being analyzed there exists a balanced c.c.s. $H^p \div h_\alpha^p(t, x, u_2, \ldots, u_m, \varepsilon)$.

2. The approach to be used for proving Theorem 1.1 was first applied for nonantagonistic differential games in the case of pure strategies (for system (1.1) with a separable right-hand side) in /3/ and was developed in /4/. In /5/ this approach was used to prove the existence theorem for a balanced coalition mixed strategy. Later on we require certain results from the theory of antagonistic differential games /1,2/. Consider an antagonistic game whose dynamics are described by Eq. (1.1); the first player's role is played by the i-th player whose problem is to minimize a quantity $\sigma_i(x[T])$, while the second player's role is played by a coalition of the other $m-1$ players whose goal is the opposite. The first player, in forming his own control, uses the counterstrategy $U_i \div u_i(t, x, u_{i+1}, \ldots, u_m, \varepsilon)$, while the second player uses a collection of counterstrategies and pure strategies

$$V_i \div \{u_1(t, x, u_i, \varepsilon), \ldots, u_{i-1}(t, x, u_i, \varepsilon), u_{i+1}(t, x, \varepsilon), \ldots, u_m(t, x, \varepsilon)\}.$$

It can be verified that such classes of player behavior are consistent, which enables us to combine the problems of the first and second players into one differential game /1/. It can be shown that this differential game (we denote it Γ_i) has a value and a universal saddle point /2/. The proofs of the existence theorems for the differential game's value are based on well-known constructions /1/ and rely essentially on the fact of the presence of a saddle point in the small game in corresponding player behavior classes.

For the game Γ_i the condition for the presence of a saddle point in the small game is

$$\begin{gathered}\min_{u_i(\cdot)} \max_{u_1(\cdot), \ldots, u_{i-1}(\cdot), u_{i+1}, \ldots, u_m} l' g(t, x, u(\cdot)) = \\ \max_{u_1(\cdot), \ldots, u_{i-1}(\cdot), u_{i+1}, \ldots, u_m} \min_{u_i(\cdot)} l' g(t, x, u(\cdot)), \quad \forall l \in R^n \\ g(t, x, u(\cdot)) = g(t, x, u_1(\cdot), \ldots, u_{i-1}(\cdot), u_i(\cdot), u_{i+1}, \ldots, \\ u_m) = f(t, x, u_1(u_i), \ldots, u_{i+1}(u_i), u_i(u_{i+1}, \ldots, u_m), \\ u_{i+1}, \ldots, u_m)\end{gathered} \tag{2.1}$$

where the minimum is computed over the functions $u_i(u_{i+1}, \ldots, u_m)$ and the maximum over the functions $u_1(u_i), \ldots, u_{i-1}(u_i)$ and the vectors $u_{i+1}, \ldots, u_m$. We omit the verification of condition (2.1)'s fulfillment. We just note that this condition is consistent with the definitions of stable sets and of strategies extremal to them, given in Sect.3. We denote the value of game Γ_i by $\gamma_i(t_*, x_*)$ and the pair of strategies forming the universal saddle point by

$$U_i^\circ \div u_{i0}(t, x, u_{i+1}, \ldots, u_m, \varepsilon), \quad V_i^\circ \div \{u_{10}^{(i)}(t, x, u_i, \varepsilon), \ldots, u_{i-1,0}^{(i)}(t, x, u_i, \varepsilon), u_{i+1,0}^{(i)}(t, x, \varepsilon), \ldots, u_{m0}^{(i)}(t, x, \varepsilon)\} \tag{2.2}$$

By the definition of the universal saddle point we have

$$\max_{x[\cdot]} \sigma_i(x[T, t_*, x_*, U_i^\circ, V_i]) \leqslant \sigma_i(x[T, t_*, x_*, U_i^\circ, V_i^\circ]) = \gamma_i(t_*, x_*) \leqslant \min_{x[\cdot]} \sigma_i(x[T, t_*, x_*, U_i, V_i^\circ]) \tag{2.3}$$

for any position $(t_*, x_*) \in G$ and any U_i and V_i.

Proof. Consider the following c.c.s. $H^* \div h_\alpha^*(t, x, u_2, \ldots, u_m, \varepsilon)$:

$$h_0^*(t, x, u_2, \ldots, u_m, \varepsilon) = \{u_{10}(t, x, u_2, \ldots, u_m, \varepsilon), \ldots, u_{m-1,0}(t, x, u_m, \varepsilon), u_{m0}(t, x, \varepsilon)\} \tag{2.4}$$

$$h_i^*(t, x, u_2, \ldots, u_m, \varepsilon) = \{u_{10}^{(i)}(t, x, u_i, \varepsilon), \ldots, u_{i-1,0}^{(i)}(t, x, u_i, \varepsilon), u_{i+1,0}^{(i)}(t, x, \varepsilon), \ldots, u_{m0}^{(i)}(t, x, \varepsilon)\}, \quad i = 1, \ldots, m$$

where all the functions comprising the collection in (2.4) have been defined in (2.2). Let us show that the c.c.s. H^* is balanced. Indeed, for any position $(\vartheta, \xi) \in G$ and any number $i = 1, 2, \ldots, m$, on the strength of (2.2), (2.3) and of the definition of motions we have

$$\min_{x[\cdot]} \sigma_i(x[T, \vartheta, \xi, H^*, \alpha_{\vartheta'}]) = \min_{x[\cdot]} \sigma_i(x[T, \vartheta, \xi, V_i^\circ]) =$$
$$\sigma_i(x[T, \vartheta, \xi, U_i^\circ, V_i^\circ]) = \gamma_i(\vartheta, \xi) \geqslant \max_{x[\cdot]} \sigma_i(x[T, \vartheta, \xi, U_i^\circ, V_i^*]) =$$
$$\max_{x[\cdot]} \sigma_i(x[T, \vartheta, \xi, H^*, \alpha^\circ])$$
$$V_i^* = \{u_{10}(t, x, u_2, \ldots, u_m, \varepsilon), \ldots, u_{i-1,0}(t, x, u_i, \ldots$$
$$\ldots, u_m, \varepsilon), u_{i+1,0}(t, x, u_{i+2}, \ldots, u_m, \varepsilon), \ldots, u_{m0}(t, x, \varepsilon)\}$$

which proves the balance of the c.c.s. H^*. A balanced c.c.s. has the following property which we state as a theorem.

Theorem 2.1. If a c.c.s. $H \div h_\alpha(t, x, u_2, \ldots, u_m, \varepsilon)$ is balanced, then for any initial position $(t_*, x_*) \in G$ the inequalities

$$\gamma_i(t, x[t]) \geqslant \sigma_i(x[T]), \; i = 1, 2, \ldots, m, \; t \in [t_0, T] \tag{2.5}$$

are fulfilled for any motion $x[\cdot] \in X(t_*, x_*, H, \alpha^\circ)$.

The theorem's proof is based on the fact that if an equality opposite to (2.5) is fulfilled for some i, some $\tau \in [t_*, T]$ and some motion $x^*[\cdot] \in X(t_*, x_*, H, \alpha^\circ)$, then the i-th player, adhereing to the strategy U_i° of (2.2), guarantees himself a result better than $\sigma_i(x^*[T])$, which contradicts the balance of c.c.s. H.

3. As follows from Sect.2, the c.c.s. H^* is determined on the basis of universal saddle points in the games Γ_i, which can be determined by different methods (see /2/). Below we examine *a method of constructing the universal saddle points which uses constructions close* to those applied (*) for the case of pure strategies (see /5/ as well). Suppose that the value functions $\gamma_i(t, x)$ are known for each of the games Γ_i $(i = 1, \ldots, m)$. Under the assumptions *made these functions are continuous in* G. We fix $\varepsilon > 0$ and we consider level surfaces of the form $\gamma_i(t, x) = k\varepsilon$, k is an integer. We denote

$$W_{i\varepsilon}^k = \{(t, x) \in G : \gamma_i(t, x) \leqslant k\varepsilon\} \tag{3.1}$$
$$Z_{i\varepsilon}^k = \{(t, x) \in G : \gamma_i(t, x) \geqslant k\varepsilon\}$$
$$S_{i\varepsilon}^k = W_{i\varepsilon}^k \setminus W_{i\varepsilon}^{k-1}, \quad R_{i\varepsilon}^k = Z_{i\varepsilon}^k \setminus Z_{i\varepsilon}^{k+1}$$

The sections of the sets introduced, by the hyperplane $t = \tau$, are denoted $W_{i\varepsilon}^k(\tau)$, $Z_{i\varepsilon}^k(\tau)$, etc. The family of nonintersecting sets $S_{i\varepsilon}^k$ $(R_{i\varepsilon}^k)$ forms a partitioning of set G, which is finite by virtue of the continuity of $\gamma_i(t, x)$. The sets $W_{i\varepsilon}^k$ are u_i^*-stable, while the sets $Z_{i\varepsilon}^k$ are v_i^*-stable, in the sense of the following definitions which are analogs of the definitions of u_*-stable and v_*-stable sets given in /1/.

A set W in the space $\{t, x\}$ is said to be u_i^*-stable if for any countercontrols $u_1(u_i)$, $\ldots, u_{i-1}(u_i)$ and for constant vectors $u_{i+1}, \ldots, u_m$, position $(t_*, x_*) \in W$ and number $t^* \in (t_*, T]$ there exists a motion $x(t)$ which is a solution of the contingency equation

$$x^\cdot(t) \in \mathrm{co}\,[f : f = (t, x, u_1(u_i), \ldots, u_{i-1}(u_i), u_i, u_{i+1}, \ldots, u_m), u_i \in P_i]$$

and satisfies the condition $(t^*, x(t^*)) \in W$. A set W in the space $\{t, x\}$ is said to be v_i^*-stable if for any countercontrol $u_i(u_{i+1}, \ldots, u_m)$, position $(t_*, x_*) \in W$ and number $t^* \in (t_*, T]$ there exists a motion $x(t)$ which is a solution of the contingency equation

$$x^\cdot(t) \in \mathrm{co}\,[f : f = f(t, x, u_1, \ldots, u_{i-1}, u_i(u_{i+1}, \ldots, u_m),$$
$$u_{i+1}, \ldots, u_m), u_j \in P_j, j = 1, 2, \ldots, m, j \neq i]$$

and satisfies the condition $(t^*, x(t^*)) \in W$.

In game Γ_i we define in the way following the first player's counterstrategy $U_i^e \div u_i^e(t, x, u_{i+1}, \ldots, u_m, \varepsilon)$ extremal to the family of sets $\{W_{i\varepsilon}^k\}$. With the triple $(t, x) \in G$, $\varepsilon > 0$ we associate an integer l_i such that $(t, x) \in S_{i\varepsilon}^{l_i}$. If here the set $W_{i\varepsilon}^{l_i-1}(t)$ is nonempty, then in it we find a point $w_i^{(1)}(t, x, \varepsilon)$ closest (one of the closest in the case of nonuniqueness) in the

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*) Kononenko A.F., Mathematical analysis methods for dynamic systems with a hierarcical system of controls. Dissertation at the competition for the academic degree of Doctor of Physico-Mathematical Sciences. Moscow: Computing Center of the Academy of Sciences of the USSR, 1979.

Euclidean metric to point x. Denoting $s_i^{(1)} = x - w_i^{(1)}(t, x, \varepsilon)$, we find $u_i^e(t, x, u_{i+1}, \ldots, u_m, \varepsilon)$ from the condition

$$\min_{u_i} \max_{u_1, \ldots, u_{i-1}} s_i^{(1)'} f(t, x, u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_m) = \tag{3.2}$$
$$\max_{u_1, \ldots, u_{i-1}} s_i^{(1)'} f(t, x, u_1, \ldots, u_{i-1}, u_i^e(t, x, u_{i+1}, \ldots, u_m, \varepsilon), u_{i+1}, \ldots, u_m)$$

If, however, $W_{i\varepsilon}^{l_i-1}(t) = \varnothing$, then as $u_i^e(t, x, u_{i+1}, \ldots, u_m, \varepsilon)$ we can choose any vector $u_i \in P_i$. Further, in game Γ_i we define a collection of counterstrategies and pure strategies of the second player, extremal to the family of sets $\{Z_{i\varepsilon}^k\}$

$$V_i^e \div \{u_1^{(i)e}(t, x, u_i, \varepsilon), \ldots, u_{i-1}^{(i)e}(t, x, u_i, \varepsilon), u_{i+1}^{(i)e}(t, x, \varepsilon), \ldots, u_m^{(i)e}(t, x, \varepsilon)\}$$

With the triple $(t, x) \in G$, $\varepsilon > 0$ we associate an integer k_i such that $(t, x) \in R_{i\varepsilon}^{k_i}$. If here the set $Z_{i\varepsilon}^{k_i+1}(t)$ is nonempty, then in it we find a point $w_i^{(2)}(t, x, \varepsilon)$ closest (one of the closest) to point x. Denoting $s_i^{(2)} = x - w_i^{(2)}(t, x, \varepsilon)$, we find $u_{i+1}^{(i)e}(t, x, \varepsilon), \ldots, u_m^{(i)e}(t, x, \varepsilon)$, as well as $u_1^{(i)e}(t, x, u_i, \varepsilon), \ldots, u_{i-1}^{(i)e}(t, x, u_i, \varepsilon)$, respectively from the conditions

$$\min_{u_{i+1}, \ldots, u_m} \max_{u_i} \min_{u_1, \ldots, u_{i-1}} s_i^{(2)'} f(t, x, u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_m) = \tag{3.3}$$
$$\max_{u_i} \min_{u_1, \ldots, u_{i-1}} s_i^{(2)'} f(t, x, u_1, \ldots, u_{i-1}, u_i, u_{i+1}^{(i)e}(t, x, \varepsilon), \ldots, u_m^{(i)e}(t, x, \varepsilon))$$

$$\min_{u_1, \ldots, u_{i-1}} s_i^{(2)'} f(t, x, u_1, \ldots, u_{i-1}, u_i, u_{i+1}^{(i)e}(t, x, \varepsilon), \ldots, u_m^{(i)e}(t, x, \varepsilon)) = \tag{3.4}$$
$$s_i^{(2)'} f(t, x, u_1^{(i)e}(t, x, u_i, \varepsilon), \ldots, u_{i-1}^{(i)e}(t, x, u_i, \varepsilon), u_i, u_{i+1}^{(i)e}(t, x, \varepsilon), \ldots, u_m^{(i)e}(t, x, \varepsilon))$$

If, however, $Z_{i\varepsilon}^{(k_i+1)}(t) = \varnothing$, then we can choose any vectors $u_1 \in P_1, \ldots, u_{i-1} \in P_{i-1}$, $u_{i+1} \in P_{i+1}, \ldots, u_m \in P_m$. We note that from conditions (3.2) and (3.4) the functions $u_i^e(t, x, u_{i+1}, \ldots, u_m, \varepsilon)$ and $u_1^{(i)e}(t, x, u_i, \varepsilon), \ldots, u_{i-1}^{(i)e}(t, x, u_i, \varepsilon)$ can be chosen as Borel-measurable relative to $u_{i+1}, \ldots, u_m$ and to u_i.

The first player's extremal counterstrategy U_i^e and the second player's collection of extremal counterstrategies and pure strategies V_i^e constructed in this manner exactly form the universal saddle point in game Γ_i.

4. As an example we consider a system described by the differential equation

$$d^2\eta / dt^2 = L(\alpha_1) F^{(1)} + L(\alpha_2) F^{(2)} + L(\alpha_3) F^{(3)} \tag{4.1}$$
$$\eta = \begin{Vmatrix} \eta_1 \\ \eta_2 \end{Vmatrix}, \quad L(\alpha_i) = \begin{Vmatrix} \cos\alpha_i & -\sin\alpha_i \\ \sin\alpha_i & \cos\alpha_i \end{Vmatrix}, \quad F^{(i)} = \begin{Vmatrix} F_1^{(i)} \\ F_2^{(i)} \end{Vmatrix}$$

The scalars α_i and the vectors $F^{(i)}$ are controls at the disposal of three players: the first player chooses α_1 and α_2, the second chooses α_3 and $F^{(1)}$, and the third chooses $F^{(2)}$ and $F^{(3)}$, i.e.

$$u_1 = \begin{Vmatrix} \alpha_1 \\ \alpha_2 \end{Vmatrix}, \quad u_2 = \begin{Vmatrix} \alpha_3 \\ F^{(1)} \end{Vmatrix}, \quad u_3 = \begin{Vmatrix} F^{(2)} \\ F^{(3)} \end{Vmatrix}$$

The constraints on the controls are

$$\| F^{(i)} \|^2 = (F_1^{(i)})^2 + (F_2^{(i)})^2 \leqslant 1, \quad |\alpha_i| \leqslant \beta, \quad 0 < \beta < \frac{\pi}{2} \tag{4.2}$$

The initial conditions $\eta(0) = \eta^\circ$, $\eta^\cdot(0) = \eta^{\cdot\circ}$ and the game termination instant T are specified. Player i strives to minimize the quantity

$$\sigma_i(\eta(T)) = \| \eta(T) - a^{(i)} \| \tag{4.3}$$

where $a^{(i)}$ are certain fixed points in the plane (η_1, η_2). Equation (4.1) can be interpreted as the equation of motion of a material point of unit mass in plane (η_1, η_2) under the action of a force formed by the three players. The third player selects two control-forces $F^{(2)}$ and $F^{(3)}$. The second player, knowing the selection of control $F^{(3)}$, turns it by an angle α_3 (it is agreed that a positive angle corresponds to a counterclockwise turn) and, in addition, selects a control-force $F^{(1)}$. The first player, knowing the other players' selections, chooses the angles α_1 and α_2 by which he turns the controls $F^{(1)}$ and $F^{(2)}$. The i-the player's goal is to take the material point as close as possible to the point $a^{(i)}$ at the instant $t = T$.

Setting $y_1 = \eta_1$, $y_2 = \eta_2$, $y_3 = \eta_1^\cdot$, $y_4 = \eta_2^\cdot$ in system (4.1) and making the changes of variables $x_1 = y_1 + (T - t) y_3$, $x_2 = y_2 + (T - t) y_4$, $x_3 = y_3$, $x_4 = y_4$, we obtain a system whose first two equations are

$$\begin{aligned} x_1^{\cdot} &= (T-t)\, f_1(\alpha_1, \alpha_2, \alpha_3, F^{(1)}, F^{(2)}, F^{(3)}) \\ x_2^{\cdot} &= (T-t)\, f_2(\alpha_1, \alpha_2, \alpha_3, F^{(1)}, F^{(2)}, F^{(3)}) \end{aligned} \tag{4.4}$$

where f_1 and f_2 are the projections of the vector on the right-hand side of Eq.(4.1) onto the axes of η_1 and η_2. Further we examine the truncated system (4.4) with the indices

$$\sigma_i(x[T]) = \| \{x[T]\}_2 - a^{(i)} \|, \quad \{x\}_2 = \left\| \begin{matrix} x_1 \\ x_2 \end{matrix} \right\| \tag{4.5}$$

To construct the c.c.s. H^* it is necessary to construct the collections h_0^* and h_1^* of (2.4). The components of these collections are (we omit the arguments for brevity)

$$u_{10} = \left\| \begin{matrix} \varphi(\omega_{11}) \\ \varphi(\omega_{12}) \end{matrix} \right\|, \quad u_{20} = \left\| \begin{matrix} \varphi(\omega_{23}) \\ -p^{(2)} \end{matrix} \right\|, \quad u_{30} = \left\| \begin{matrix} -p^{(3)} \\ -p^{(3)} \end{matrix} \right\| \tag{4.6}$$

$$u_{20}^{(1)} = \left\| \begin{matrix} 0 \\ p^{(1)} \end{matrix} \right\|, \quad u_{30}^{(1)} = \left\| \begin{matrix} p^{(1)} \\ p^{(1)} \end{matrix} \right\|, \quad u_{10}^{(2)} = \left\| \begin{matrix} \psi(\omega_{22}) \\ 0 \end{matrix} \right\|$$

$$u_{30}^{(2)} = \left\| \begin{matrix} p^{(2)} \\ p^{(2)} \end{matrix} \right\|, \quad u_{10}^{(3)} = \left\| \begin{matrix} 0 \\ \psi(\omega_{32}) \end{matrix} \right\|, \quad u_{20}^{(3)} = \left\| \begin{matrix} \psi(\omega_{33}) \\ p^{(3)} \end{matrix} \right\|$$

$$p^{(i)} = \frac{\{x\}_2 - a^{(i)}}{\| \{x\}_2 - a^{(i)} \|}, \quad \varphi(\omega_{ij}) = \begin{cases} \pm\beta & , \ \omega_{ij} = \pi \\ \beta \operatorname{sign} \omega_{ij} & , \ \pi > |\omega_{ij}| \geqslant \beta \\ \omega_{ij} & , \ |\omega_{ij}| < \beta \end{cases}$$

$$\psi(\omega_{ij}) = \begin{cases} \pm\beta & , \ \omega_{ij} = 0 \\ -\beta \operatorname{sign} \omega_{ij} & , \ 0 < |\omega_{ij}| \leqslant \pi - \beta \\ -\omega_{ij} + \pi \operatorname{sign} \omega_{ij}, & \pi - \beta < |\omega_{ij}| \leqslant \pi \end{cases}$$

Here $\omega_{ij} = \omega_{ij}(x)$ $(i, j = 1, 2, 3)$ is the angle between the vectors $F^{(j)}$ and $-p^{(i)}$, $-\pi < \omega_{ij}(x) \leqslant \pi$, and $\omega_{ij} > 0$ if the shortest rotation of vector $F^{(j)}$ up to coincidence with vector $-p^{(i)}$ takes place counterclockwise.

The numerical experiment was performed on a computer. The following numerical values of the problem's parameters were chosen:

$$T = 2.2; \quad \beta = \pi/3, \quad a^{(1)} = \left\| \begin{matrix} 0 \\ -1 \end{matrix} \right\|, \quad a^{(2)} = \left\| \begin{matrix} 0 \\ 1 \end{matrix} \right\|, \quad a^{(3)} = \left\| \begin{matrix} -2 \\ 0 \end{matrix} \right\|$$

$$\left\| \begin{matrix} y_1(0) \\ y_2(0) \end{matrix} \right\| = \left\| \begin{matrix} 0.78 \\ 2.20 \end{matrix} \right\|, \quad \left\| \begin{matrix} y_1^{\cdot}(0) \\ y_2^{\cdot}(0) \end{matrix} \right\| = \left\| \begin{matrix} 0.10 \\ 1.00 \end{matrix} \right\|$$

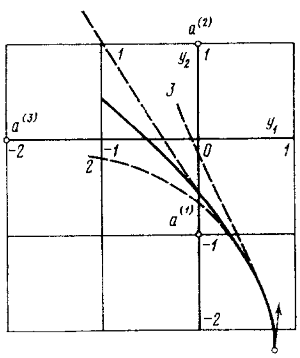

Fig.1

The Fig.1 shows the Euler polygonal lines generated by the c.c.s. H^* for a uniform partitioning step $\delta(\Delta) = 0.01$. The solid line shows the Euler polygonal line in the game without vilations. The dashed line *1* shows the Euler polygonal line in a game where the violator is the first player and the violation instant is $\vartheta = 1.2$; the digit *2* marks the polygonal line in a game where the violator is the second player and $\vartheta = 0.9$; the digit *3* marks the Euler polygonal line in a game where the violator is the third player and $\vartheta = 0.6$. Here the violators choose control methods which were the best under the counteractions of the other players. We see that the violators essentially increase the distances from their own target points, as compared with those they have in the violation-free game. We remark that in the Fig.1 we have shown only individual Euler polygonal lines from the appropriate sheaves.

The author thanks A.I. Subbotin for attention to the work and for discussing the results, as well as V.L. Turova for assistance with the numerical experiment.

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Translated by N.H.C.